proved, and therefore the eigenvalues obtained from the two different formulations must satisfy the above equations if the linear theory is applied to the situation where it is valid. No additional demonstration of equivalence is necessary, although the eigenvalues obtained from the two formulations are, in general, not the same in the region where the linear theory is invalid.

All previous workers, including Heisenberg (1924) who used the temporal formulation, need not be embarrassed. They should be congratulated because they used the theory which is capable of predicting the correct condition for the onset of instability and also the correct initial amplification rate which is simply related to the spatial growth rate by the Gaster transformation. Those workers who believe that spatial formulation of the linear stability theory is superior should be reminded of its shortcomings. Note that only one Fourier component of the disturbance is considered in either the temporal or spatial formulation of the theory. However, a spatially bounded but temporarily growing arbitrary disturbance $f(x, t)$ can be constructed by the Fourier super position

$$f(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) e^{i(\alpha x - \omega t)} \, d\alpha$$

where

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x, t) e^{-i(\alpha x + \omega t)} \, dx$$

Since the Orr-Sommerfeld equation is linear, it can be shown easily that the eigenvalue, which gives the stability condition, for the above general disturbance is identical to that for a single Fourier component. Thus the usual normal mode approach of the stability theory is justifiable for the temporally growing disturbances. On the other hand, because of the exponential factor $\exp(-\alpha_i x)$, the Fourier integral of the spatially growing disturbance does not exist unless $\alpha_i \rightarrow 0$. Thus the normal mode approach is not justifiable for the case of spatial formulation except when $\alpha_i \rightarrow 0$. However, when $\alpha_i \rightarrow 0$, two formulations are equivalent as was shown in the previous paragraphs. In short, the two formulations are equivalent if the linear theory is applied to where it is valid. Finally, two formulations are truely nonequivalent when nonlinear effects are considered (Agrawal and Lin, 1975).

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A Mixture Theory Formulation for Particulate Sedimentation

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The sedimentation, or settling, of one substance in solid particulate form through a second substance in liquid or gaseous form plays an important part in many chemical engineering processes. A large number of examples are cited by Zenz and Othmer (1960). Sedimentation is also of importance in medicine, where erythrocyte sedimentation has become a standard clinical test, and in meteorology.

Previous work on analyzing the sedimentation process typically has concentrated on developing empirical expressions for the terminal settling velocity of the particles in terms of the properties of the particles and fluid, and the volumetric concentration of the particles (Barnea and Mizrahi, 1973).

In the present work, the analysis of particulate sedimentation is approached within the framework of the continuum theory of mixtures (Bowen and Wiese, 1969; Bedford and Ingram, 1971) by regarding the particles and fluid as superimposed continua. The equations obtained are applicable to the analysis of transient one-dimensional sedimentation and, when specialized to the steady sedimentation of particles with uniform particle concentration, lend additional insight into the terminal settling velocity problem.

DEVELOPMENT

The problem considered is the sedimentation under gravity of solid particles of uniform composition, size, and shape through a fluid held in a container of constant cross section (Figure 1). The x axis denotes position, with the positive direction downward.

Regarding the particles and fluid as two superimposed continua, in the absence of reactions, dissolution, or other mass transfer processes, each constituent must satisfy the usual one-dimensional conservation of mass equation

$$\frac{\partial \hat{\rho}_p}{\partial t} + \frac{\partial}{\partial x} (\hat{\rho}_p U_p) = 0 \qquad (1)$$

$$\frac{\partial \hat{\rho}_f}{\partial t} + \frac{\partial}{\partial x} (\hat{\rho}_f U_f) = 0 \qquad (2)$$

Writing the partial mass densities $\hat{\rho}_p$ and $\hat{\rho}_f$ in terms of mass densities and volumetric concentrations as $\hat{\rho}_p = \phi_p \rho_p$ and $\hat{\rho}_f = \phi_f \rho_f$, and assuming incompressibility of the individual particles and the fluid so that ρ_p and ρ_f are constant, we can write Equations (1) and (2) as

$$\frac{\partial \phi_p}{\partial t} + \frac{\partial}{\partial x} (\phi_p U_p) = 0 \qquad (3)$$

$$\frac{\partial \phi_f}{\partial t} + \frac{\partial}{\partial x} (\phi_f U_f) = 0 \qquad (4)$$

Also, note that

$$\phi_p + \phi_f = 1 \qquad (5)$$

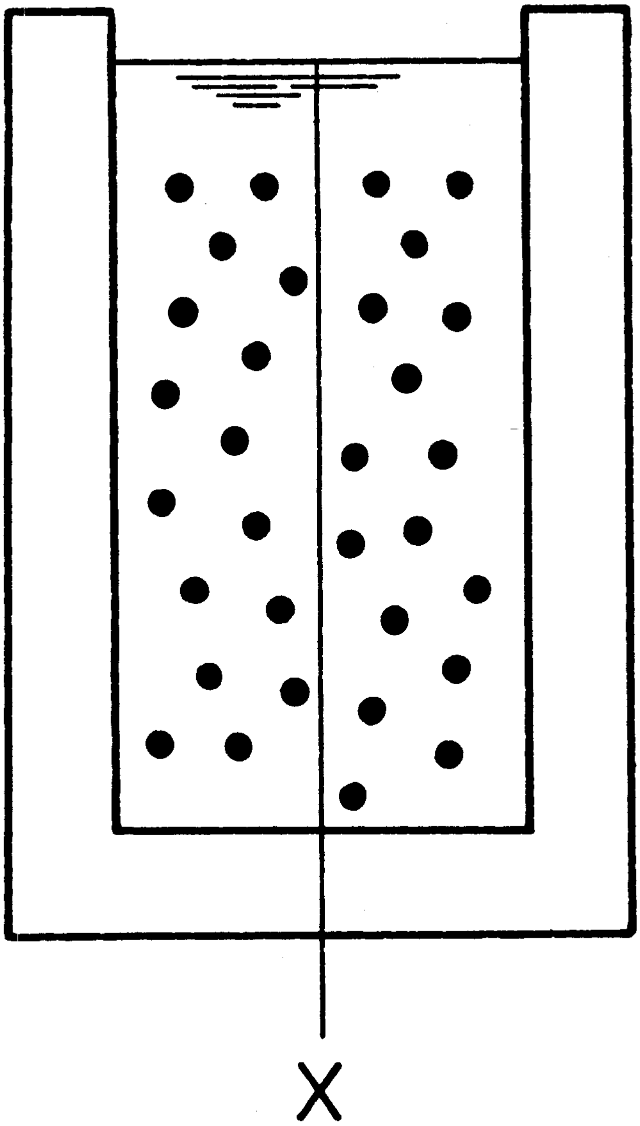

Fig. 1. Sedimentation configuration.

The total force in the x direction on a single particle including drag, weight, and buoyancy is

$$f = -\alpha(\phi_p, U_p - U_f) + \rho_p V g - \rho V g \quad (6)$$

The dependence of the drag force α on the volumetric concentration and the relative velocity has been indicated in Equation (6). Determination of the dependence of the drag on these variables and on the fluid and particle properties has been the principal objective of previous experimental and theoretical studies on sedimentation. (Barnea and Mizrahi, 1973.) For very dilute particle concentrations and very small sedimentation velocities, Stokes' law can be used for α.

Note that in writing the buoyancy force, the mixture mass density $\rho = \hat{\rho}_p + \hat{\rho}_f$ has been used rather than the fluid density after Richardson and Meikle (1961).

The force on the particles per unit volume of mixture is then $(\phi_p/V)f$, so that the linear momentum equation for the particle continuum can be written

$$\hat{\rho}_p a_p = -(\phi_p/V)\,\alpha(\phi_p, U_p - U_f) + \phi_p g(\rho_p - \rho) \quad (7)$$

where the particle acceleration is

$$a_p = \frac{\partial U_p}{\partial t} + U_p \frac{\partial U_p}{\partial x} \quad (8)$$

Equations (3), (4), and (5) and Equation (7) with Equation (8) provide four equations in the four variables ϕ_p, ϕ_f, U_p, U_f.

Adding Equations (3) and (4) and using Equation (5), we get

$$\frac{\partial}{\partial x}(\phi_p U_p + \phi_f U_f) = 0 \quad (9)$$

The term $\phi_p U_p + \phi_f U_f$ is the volume of mixture passing a point x per unit time per unit area. For the sedimentation configuration of Figure 1, with velocities measured relative to the container, that volume will be zero, so that integration of Equation (9) yields

$$U_f = -\frac{\phi_p}{\phi_f} U_p \quad (10)$$

Using Equations (5) and (10) to eliminate ϕ_f and U_f from Equation (7) with Equation (8), we get

$$\rho_p \left(\frac{\partial U_p}{\partial t} + U_p \frac{\partial U_p}{\partial x}\right) = -(1/V)\,\alpha(\phi_p, [1/(1 = \phi_p)]U_p) + g(\rho_p - \rho_f)(1 - \phi_p) \quad (11)$$

so that Equations (3) and (11) provide two equations in the two variables ϕ_p and U_p. Equations (3) and (11) are not linear but are quasilinear and, together with an experimental or empirical relation for α, will provide numerical solutions for transient sedimentation problems of practical interest.

For steady sedimentation of uniformly distributed particles, Equation (3) is satisfied identically, and Equation (11) becomes

$$\alpha(\phi_p, [1/(1-\phi_p)]U_p) = Vg(\rho_p - \rho_f)(1 - \phi_p) \quad (12)$$

which merely indicates that the drag and buoyancy forces are balanced by the weight.

Equation (12) notes that the particle drag is governed not by the particle velocity relative to the container U_p but by the velocity relative to the fluid

$$U_p - U_f = [1/(1 - \phi_p)]U_p \quad (13)$$

This effect is not important for small particle concentrations but is of major importance for higher concentrations. This fact has not always been recognized in previous sedimentation research and, even when recognized, has often been included empirically, rather than by the simple Equation (13) (Barnea and Mizrahi, 1973).

NOTATION

a_p = particle acceleration
f = force on a single particle
g = acceleration of gravity
t = time
U_f = fluid velocity
U_p = particle velocity
V = volume of a single particle
x = distance

Greek Letters

α = drag force on a single particle
ρ = mass per unit volume of fluid-particle mixture
ρ_f = mass per unit volume of fluid
ρ_p = mass per unit volume of a single particle
$\hat{\rho}_f$ = mass of fluid per unit volume of fluid-particle mixture
$\hat{\rho}_p$ = mass of particles per unit volume of fluid-particle mixture
ϕ_f = volume of fluid per unit volume of fluid-particle mixture

ϕ_p = volume of particles per unit volume of fluid-particle mixture

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A Unified Treatment of Drainage, Withdrawal, and Postwithdrawal Drainage with Inertial Effects

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The aim of this note is twofold: first, we attempt a unified description of the drainage, withdrawal, and postwithdrawal drainage of a liquid over a flat plate; second, we examine the question of which of two conditions that have appeared in the literature is to be employed for correctly obtaining the film thickness profiles. A knowledge of these profiles is of importance in a variety of applications, such as dipcoating, enameling, electroplating, and capillary viscometry (Tallmadge and Gutfinger, 1967).

PLATE LIFTED WITH A GENERAL VELOCITY

We choose the yz and xz planes to coincide with the initial position of the plate and the initial surface of the bath. At $t = 0$, the plate is lifted along the y axis with a moderate velocity $f(t)$, and the bath is allowed to drain freely under the action of gravity. The solution of the problem will be obtained for arbitrary $f(t)$. From this, the results for the drainage, withdrawal, and post withdrawal processes emerge as special cases for appropriate choices of the plate velocity $f(t)$.

We confine ourselves to the parallel flow region and neglect the effect of surface tension on the flow. With the assumption of laminar one-dimensional flow, permissible in this region (Groenveld, 1970), the Navier-Stokes equation in the y direction reduces to (Gutfinger and Tallmadge, 1964)

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial x^2} - g \qquad (1)$$

The velocity $v(x, t)$ of the liquid is subject to the usual conditions

$$v(x,0) = 0, \quad \frac{\partial v}{\partial x}(h,t) = 0 \quad \text{and} \quad v(0,t) = f(t) \qquad (2)$$

Taking the Laplace transform of Equation (1), we obtain

$$\frac{d^2\bar{v}}{dx^2} - \frac{s}{\nu}\bar{v} = \frac{g}{s\nu} \qquad (3)$$

where $\bar{v}(x, s)$ is required to satisfy the conditions

$$\bar{v}(0,s) = \bar{f}(s) \quad \text{and} \quad \frac{d\bar{v}}{dx} = 0 \quad \text{at} \quad x = h \qquad (4)$$

Equation (3) has the solution

$$\bar{v}(x,s) = \left[\bar{f}(s) + \frac{g}{s^2}\right] \frac{\cosh\sqrt{\frac{s}{\nu}}\,(x-h)}{\cosh\sqrt{\frac{s}{\nu}}\,h} - \frac{g}{s^2} \qquad (5)$$

The inverse of Equation (5) gives the solution of Equation (1):

$$v(x,t) = \frac{g}{2\nu}(x^2 - 2xh) + \frac{2gh^2}{\nu}\sum_{n=0}^{\infty} \frac{e^{-\alpha_n^2 \nu t/h^2}}{\alpha_n^3} \sin\frac{\alpha_n x}{h} + \frac{2\nu}{h^2}\sum_{n=0}^{\infty} \alpha_n e^{-\alpha_n^2 \nu t/h^2} \sin\frac{\alpha_n x}{h} \cdot \int_0^t f(w)\, e^{\alpha_n^2 \nu w/h^2} dw,$$

$$x > 0 \quad = f(t), \quad \text{for} \quad x = 0 \qquad (6)$$

where $\alpha_n = (n + 1/2)\pi$, and w is the supplementary variable brought in by the convolution theorem. In terms of nondimensional quantities, the flux of the entrained liquid is obtained as

$$Q = \int_0^H V(X,T)\,dX = -\frac{H^3}{3} + 2H^3 \sum_{n=0}^{\infty} \frac{e^{-\alpha_n^2 T/H^2}}{\alpha_n^4} + \frac{2}{H}\sum_{n=0}^{\infty} e^{-\alpha_n^2 T/H^2} \cdot \left[\int_0^T F(W)\, e^{\alpha_n^2 W/H^2} dW\right] \qquad (7)$$

The film thickness profile is connected to the flow rate through the equation of continuity, which yields

$$Y = \int \left(\frac{\partial Q}{\partial H}\right)_T dT + \phi(H) \qquad (8)$$

In order to evaluate the integration 'constant' $\phi(H)$,